Australian Government
Department of Defence
Defence Science and
Technology Organisation

Application of Black Scholes Complexity Concepts to Combat Modelling

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DSTO-TR-2318

ABSTRACT

Lanchester's equations are commonly used as the basis for force-on-force combat models, even if only as a metamodel for a more complex combat simulation. This report examines whether attrition is adequately modelled by such Markov processes. It shows that the distribution of historical battle casualties is consistent with that obtained when attrition is modelled as an Ito process. The additional Wiener term can be regarded as representing the impact of the wider environment on attrition rates.

RELEASE LIMITATION

Approved for public release

Published by

Joint Operations Division
DSTO Defence Science and Technology Organisation
Fairbairn Business Park Department of Defence
Canberra ACT 2600 Australia

Telephone: (02) 6265 9111
Fax: (02) 6265 2741

AR-014-581
March 2009

APPROVED FOR PUBLIC RELEASE

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Executive Summary

Lanchester's equations are commonly used as the basis for *force-on-force* combat models, even if only as a metamodel for a more complex combat model. These equations define a system with the strengths of the forces involved comprising its internal parameters. Many systems are adequately described using just their internal parameters, without consideration of any interactions between that system and its wider environment. However, it is apparent from the work on extending combat models based on Lanchester's equations to include additional parameters such as morale, spatial force dispersion and movement, that such quantities do affect attrition rates. The inclusion of additional parameters also results in additional complexity and the loss of insight that a simple model provides. ideally what is desired is a means to include the effect of the wider environment on attrition rates without also increasing the model's complexity.

The standard model for the behaviour of stock prices in time assumes they are a continuous Markov process with a constant fractional drift rate. The Black and Scholes model of stock prices treats price volatility as resulting from the action of the rest of the market on the system comprised of the one stock price being modelled. Furthermore, it does not attempt to model the processes by which the market might affect the stock price, arguing that the mechanisms are too complex to model or are not known.

Lanchester's Equations are similar to the starting point for the derivation of the Black Scholes Equation. This suggests an obvious approach for including the effect of the wider environment in the evaluation of combat attrition rates, through the addition of a Wiener process, turning Lanchester's Markov process into an Ito process.

The present work has used an existing database of historical battle results to show that the frequency distribution of battle casualties is consistent with that expected when Lanchester's equations are augmented to form an Ito Process rather than the conventional Markov Process. The additional Wiener term can be regarded as representing the impact of the wider environment on attrition rates. The shape of the casualty frequency distribution was not observed in the initial force strength distribution. This supports the contention that such distributions result from the attrition process itself and are not artefacts of the sampling or analysis procedure.

The database used, and indeed all such databases, was shown to include an inherent bias which under-represents the number of small battles. While the effect of such bias was observed, by incorporating strata sampling concepts it was possible to confine the effects of such bias into a single stratum and a small number of data points, which can then be allowed for.

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Contents

Acronyms, Abbreviations and Symbols

a,b	attrition coefficients
a_i	Independent model variables
$a(\mathrm{x,t})$	Variable drift rate
A_i	Co-ordinate transformation matrix
A	Lanchester Equation solution parameter
b_i	Dependent model variables
$b(\mathrm{x,t})$	Variable variance rate
B	Lanchester Equation solution parameter
$c..k$	Arbitrary Exponents
dQ	Heat Flow
dU	Change in Internal Energy
dW	Work done
dz_i	Wiener Process
dx/dt	Rate of change of X force strength with time
dy/dt	Rate of change of Y force strength with time
$f(\,)\ g()$	Arbitrary functions
I	Combat intensity
ln	natural logarithm
O	Model outcome
$p_i..r_i$	Arbitrary Exponents
R	Relative combat effectiveness
R^2	Coefficient of determination
S	Stock price
X	Force X
$x(t)$	Strength of X force at time t
x_j	Strength of X force of type j
x_0	Initial strength of X force
Y	Force Y
$y(t)$	Strength of Y force at time t
y_j	Strength of Y force of type j
y_0	Initial strength of Y force
y_f	Final strength of Y force
α,β	Arbitrary Exponents
δt	incremental change in time t
δx	incremental change in variable x
δz	incremental Wiener process
ε,σ	mean variance rates
μ	mean drift rate
$\Phi(\,)$	Normal distribution function
$\varphi(\,)$	Arbitrary dimensionless function
Π_j	Similarity Parameters
$[x]$	Dimension of variable x

1. Introduction

A model of combat has long been sought, arguably since the time of Sun Tze half a millennium before the Christian era. However, prior to the early Twentieth Century all models that had been developed were essentially descriptive or narrative in nature. This changed with the advent of Lanchester's model of air combat in 1914 [1], which was the first attempt to construct a mathematical model of combat. Lanchester's equations have subsequently been used as the basis for most *force-on-force* combat models, even if only as a metamodel for a more complex combat simulation [2] [3].

Lanchester's mathematical formulation of his model was incomplete. In addition to his two equations, it required processes to initiate combat, commit forces and allocate effort, arrangement of forces to carry out those decisions to facilitate a hierarchical decomposition of a battle into smaller sub-battles to which Lanchester's Equations are then applied. Extensive research to develop mathematical formulations to replace those processes has subsequently taken place [4].

Combat between two sides of strength *x(t)* and *y(t)* is generally described by simple differential equations, such as those for Lanchester's "modern" combat:

$$\begin{aligned}\frac{dx}{dt} &= -ay(t), \quad x(0) = x_0 \\ \frac{dy}{dt} &= -bx(t), \quad y(0) = y_0\end{aligned} \tag{1}$$

which may be modified to include additional combat effects [4], or additional differential equations are sometimes added to describe the interaction of other parameters [5]. Indeed, Taylor [4] has expressed the belief that regardless of how a mathematical model of combat is constructed, the attrition of engaged forces should be expressible in terms of a series of coupled differential equations, which in general terms may be written as:

$$\begin{aligned}\frac{dx_i}{dt} &= f_i\left(x_1,\ldots x_k, y_1,\ldots y_n, b_1,\ldots b_m\right) \\ \frac{dy_j}{dt} &= g_j\left(x_1,\ldots x_k, y_1,\ldots y_n, b_1,\ldots b_m\right)\end{aligned} \tag{2}$$

where *f( )* and *g()* are arbitrary functions of the independent variables x_i and y_i of sides *X* and *Y* as well as of the dependent parameters b_i.

The present work will examine the application of a common technique by which simple metamodels are developed from such systems of equations, before considering an additional approach for including the impact of the wider environment. Finally, it will compare observed casualty patterns in available historical data with a proposed attrition model.

2. Combat Metamodels

A model can be considered as a representation of an actual situation that may be used to better understand that situation. Complex phenomena often require complex models if the model's behaviour is to reproduce that of the real world. However, while such models produce reasonable agreement with real world results, they are less useful in understanding the functional dependence of the modelled quantities on the input parameters. In such cases it is useful to develop a (simpler) model of that model which, although providing lower fidelity results, is better at explaining the causes of those results. Such models are the *metamodels*.

Metamodels described by systems of equations similar to those above (equation 2) are often developed using an *ad-hoc* unstructured approach such as dimensional analysis. In recent years a systematic treatment of the process for the development of such metamodels has emerged (intermediate asymptotic approximation) that provides a degree of rigour to the undertaking [6].

While the interested reader is referred to the work by Barenblatt [6] for a comprehensive treatment of the application of similarity principles, a brief summary is given below. It must be noted that the use of dimensional analysis in the study of similarity laws is only strictly correct for self-similar problems. However, previous work by the author [3] and others [2] provides some justification in the application of this approach in combat attrition modelling.

2.1 Intermediate Asymptotics

The relationship between an outcome of the model, and a set of input variables can be written as:

$$O = f\left(a_1, \ldots a_k, b_1, \ldots b_m\right) \qquad (3)$$

where the variables $a_1,\ldots a_k$ have independent dimensions (the dimension of any of the a's cannot be expressed as a combination of the dimensions of the other a's), and the dimensions of the $b_1,\ldots b_m$ can be expressed in terms of products of powers of the dimensions of $a_1,\ldots a_k$. In general k > 0 and m > 0. In the example system of equation 2, the model outcomes are the rates of change of the strengths of the forces engaged.

The values of the $a_1,\ldots a_k$ can be independently varied, so that

$$a'_i = A_i a_i \qquad (4)$$

The dimensions of a and $b_1,\ldots b_m$ may then be represented as power monomials in the dimensions of $a_1, \ldots a_k$, for example:

$$\begin{aligned}[b_j] &= [a_1]^{p_j} \ldots [a_k]^{r_j} \\ [O] &= [a_1]^{p} \ldots [a_k]^{r}\end{aligned} \tag{5}$$

With the corresponding transformation of values:

$$\begin{aligned}b'_j &= A_1^{\,p_j} \ldots A_k^{\,r_j} b_j \\ O' &= A_1^{\,p} \ldots A_k^{\,r} O\end{aligned} \tag{6}$$

Introducing the similarity parameters:

$$\begin{aligned}\Pi_j &= \frac{b_j}{a_1^{p_j} \ldots\ldots a_k^{r_j}} \\ \Pi &= \frac{O}{a_1^{p} \ldots\ldots a_k^{r}}\end{aligned} \tag{7}$$

where the exponents of the variables are chosen so that the parameters Π and $\Pi_1,\ldots\Pi_m$ are dimensionless, enables equation (3) to be re-written as:

$$\Pi = \frac{1}{a_1^{p} \ldots a_k^{r}} f\left(a_1, \ldots a_k, \Pi_1 a_1^{p_1} \ldots a_k^{r_1}, \ldots \Pi_m a_1^{p_m} \ldots a_k^{r_m}\right) \tag{8}$$

Barenblatt [6] shows that this in turn can be re-written in terms of a function of a smaller number of dimensionless variables, leading to the relationship:

$$O = a_1^{p} \ldots a_k^{r} \varphi\left(\frac{b_1}{a_1^{p_1} \ldots a_k^{r_1}}, \ldots \frac{b_m}{a_1^{p_m} \ldots a_k^{r_m}}\right) \tag{9}$$

Furthermore, self-similar solutions correspond to cases where the values of the variables $b_1,\ldots b_m$ tend to zero or infinity. Barenblatt [6] considers three cases:

1. Type 1 metamodel.

The function φ tends to a non-zero finite limit as Π_j tends to zero or infinity. In practice this means φ can be replaced by its limiting expression, and hence *f* will be a product of power monomials whose values can be determined by dimensional analysis.

2. Type 2 metamodel.

The function φ tends to the power law asymptotic expression:

$$\varphi = \Pi_j^{\alpha} \varphi\left(\frac{\Pi_1}{\Pi_j^{\alpha_1}}, \ldots \frac{\Pi_m}{\Pi_j^{\alpha_m}}\right) \tag{10}$$

as Π_j tends to zero or infinity. The power law form of the limiting expression still leads to complete separation of variables, but with characteristic exponents which, in contrast to the type 1 metamodels, cannot all be determined by dimensional analysis.

3. Type 3 metamodel.

Power type asymptotic behaviour is not observed, and the function φ has no finite limit different from zero.

2.2 Structure

Many physical systems conform to the requirements for type 1 or type 2 metamodel approximations. Systems that produce type 3 metamodels will not be considered further. Previous work has shown that combat attrition models result in type 2 metamodels [2] [3] with governing equations of the form:

$$\begin{aligned} \frac{dx}{dt} &= -a^c y(t)^d x(t)^e, \quad x(0) = x_0 \\ \frac{dy}{dt} &= -b^f x(t)^g y(t)^h, \quad y(0) = y_0 \end{aligned} \tag{11}$$

For the cases of interest here, the metamodel can be written as a product of power law monomials, where some exponents may be unknown, of the internal parameters used to describe the system's dynamics.

3. Including the Environment

It was noted above that metamodels of the intermediate asymptotic form use the system's internal parameters to describe its dynamic behaviour. Many systems are adequately described using just their internal parameters, without consideration of any interactions between that system and its wider environment. This is a standard approach based on the assumption that the system can be defined in such a way that changes in its own internal parameters do not involve external interactions. One such example system is the motion of the earth and its moon. For most purposes only the properties of the earth and moon need to be considered. The gravitational effect of all other masses (its environment) can be ignored.

However, there are also many systems where the interaction between the system and its environment cannot be neglected. In such cases, either a more extensive definition of the system must be used where the new system's dynamics are independent of its environment, or a simple mechanism for the interaction between the system and environment developed. One such example system is the First law of Thermodynamics:

$$dU = dQ + dW \tag{12}$$

where the environment is treated as a heat source or sink (dQ) for the system's internal energy U.

The first approach is commonly used in combat modelling, which has been reviewed by Taylor [4], and adopted in most current generation military combat simulations [7]. This has the drawback that these system definitions are generally so complex as to require their own metamodels to provide insight and so do not simplify the problem.

The second approach has also been used, to a lesser degree, in combat modelling where it represents losses due to non-combat processes such as disease, accident or reinforcements [8] and typically produces equations of the form:

$$\begin{aligned} \frac{dx}{dt} &= P - ay - cx, \quad x(0) = x_0 \\ \frac{dy}{dt} &= Q - bx - dy, \quad y(0) = y_0 \end{aligned} \tag{13}$$

More complex variants of Lanchester's Equations generally produce better agreement between theory and the trends exhibited by historical data. However, the effect of the environment on combat outcomes is poorly understood [9].

In order to avoid developing complex models of the mechanisms for interaction between the system and its environment, which are often not understood anyway, a simple model of the effect of interaction with a complex environment is needed. This is not the oxymoron that it first appears to be. Such approaches have been used in financial modelling for some time and have their origin in the study of Brownian motion as a stochastic process.

4. Stochastic Processes

This section follows Hull's approach [10]. A stochastic process is described by a variable whose value changes in time in an uncertain way. Such processes can be discrete, when the variable's value can only change at specified fixed points, or continuous when the value can change at any time. Stochastic processes may also take continuous values, when the underlying variable can take any value within a specified range, or discrete values where only certain specified values are allowed.

A Markov process is a particular type of stochastic process where only the current value of a variable is relevant for predicting its future evolution. A continuous time, discrete value Markov process has been demonstrated to produce a stochastic attrition model analogous to Lanchester's deterministic equations. Most modern combat simulations use Markov

processes to describe attrition. The stochastic theory of attrition has been comprehensively explored by a number of workers and is readily accessible [11].

A basic Wiener process is a sub-type of Markov process in which changes to the value of the underlying variable during successive time intervals are normally distributed with a mean of zero and a variance of "*1 per time interval*". Brownian motion is a Wiener process. A variable z follows a Wiener process if it has the following two properties:

- Property 1: The change δz during a small time period δt is: $\delta z = \varepsilon\sqrt{\delta t}$ where ε is a random number from a standardised normal distribution ϕ() with a mean of 0 and a variance of 1.
- Property 2: The values of δz for any two different intervals δt are independent.

A generalised Wiener process is similar to the basic Wiener process, with its drift rate of 0 and variance rate of 1, in that its drift rate and variance rate may change with time. Such a process for a variable x has a stochastic differential of the form:

$$dx = a\,dt + b\,dz \tag{14}$$

where a and b are constants and z is a basic Wiener process above.

Finally, an Ito process is a generalised Wiener process in which the parameters a and b are functions of the underlying variable x and time t.

$$\begin{aligned} dx &= a(x,t)dt + b(x,t)dz \\ \delta x &= a(x,t)\delta t + b(x,t)\varepsilon\sqrt{\delta t} \end{aligned} \tag{15}$$

with a drift rate of $a(x,t)$ and a variance rate of $b(x,t)^2$. The stochastic processes above have only considered systems with one stochastic variable. There is not an inherent limitation. Considering a system with two Markov variables x and y, it is straightforward to see that Lanchester's Equations represent a special case of an Ito process with two coupled stochastic variables having variance rates equal to 0 and drift rates defined by equation 1 above.

4.1 Black Scholes Model

The standard model for the behaviour of stock prices in time assumes they are a continuous Markov process with a constant fractional drift rate. Thus the stock price S with a drift rate of μ is described by:

$$\frac{dS}{S} = \mu dt \tag{16}$$

which is a system with one internal variable S and one internal parameter μ. In practice, stock prices exhibit considerable stochastic volatility. The Black and Scholes model of stock

prices [12] treats the volatility as resulting from the action of the rest of the market on the system composed of the one stock price being modelled. Furthermore, it does not attempt to model the processes by which the market might affect the stock price, arguing that the mechanisms are too complex to model or are not known. However, by invoking the central limit theorem they believe that the effect of this interaction can be modelled even when the mechanisms are not known. This modifies equation 16 to an Ito process as:

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{17}$$

The first term in this expression results from the interaction between the system's underlying parameters, while the second results from the action of the wider environment on the system. The analytic solution for equation 16 is a single exponential distribution. Adding the Wiener term (equation 17) modifies this solution to a log-normal distribution. The similarity between this equation and the Lanchester Equations (equation 1) is clear. This suggests an obvious solution to the question of how to include the effect of the wider environment in the evaluation of combat attrition rates. Following the Black Scholes approach:

$$\begin{aligned} \frac{dx}{y} &= -adt + \sigma dz_2, \quad x(0) = x_0 \\ \frac{dy}{x} &= -bdt + \varepsilon dz_1, \quad y(0) = y_0 \end{aligned} \tag{18}$$

and σ and ε are measures of the environment's contribution, while the z_i are Wiener processes which may be independent.

The analytic solution to Lanchester's Equations (equation 1) are monotonically decreasing functions with exponential components, albeit more complex than for the stock price model above:

$$\begin{aligned} x &= \sqrt{R}\left(-Ae^{It} + Be^{-It}\right) \\ y &= \left(Ae^{It} + Be^{-It}\right) \end{aligned} \tag{19}$$

where:

$$\begin{aligned} I &= \sqrt{ab}, \quad A = \frac{1}{2}\left(y_0 - \frac{x_0}{\sqrt{R}}\right) \\ R &= \frac{a}{b}, \quad B = \frac{1}{2}\left(y_0 + \frac{x_0}{\sqrt{R}}\right) \end{aligned} \tag{20}$$

This solution for both x and y reduces to a single exponential when the initial force ratio (x_0/y_0) equals the root of the relative effectiveness (R). The magnitude of A is a measure of how evenly the forces are matched, where A = 0 represents neither side having an

advantage and the combat is even. Provided $|A| << |B|$ the contribution from the increasing exponential term in equation 19 will be small and can be treated as a perturbation. The leading term in this solution is then also a single exponential. The solutions can be considered as a dynamical system with a positive Maximal Lyapunov Exponent [13] that increases with $|A|$.

Provided $|A| << |B|$, the proposed attrition model (equation 18) also has a log-normal distribution as its first order solution for casualty values. Such a distribution would not be expected from force strength distributions, which are dominated by the distribution of initial values (x_0, and y_0). These are controlled by different dynamics and were first investigated by Richardson [14]. The conditions under which this approximation can be expected to hold will be examined in the next section, along with a number of issues concerning the use of historical data for comparison with combat models. A general closed form analytic solution for equation 18 is still being sought.

5. Historical Analysis

Determination of whether Lanchester's Equations (equation 1) need to be augmented by terms describing the effect of the wider combat environment, as proposed in equation 18, depends on whether behaviour that can be attributed to the effect of such terms can be found in the historical record.

5.1 Historical Data

Despite the large number of recorded battles throughout history, the number with usable data is small. Any compilation of battle data, being a subset of all battles, constitutes a sample. A useful database will have a sample of battle data that is representative of patterns observed in the population of all battles.

To validate differential models of attrition, such as Lanchester's equations, force and casualty levels for both sides intermediate to the starting and finishing quantities are required. That level of detail is rarely available and often does not exist. The author has not found a single instance where sufficient data of this type is available to carry out the examination proposed in section 4. An alternate method, using only initial and final values of engaged force strengths is developed in the next section.

Some data compilations of battles throughout recorded history exist, and have been aggregated and used in previous analyses [15]. The component databases were put together by a number of workers for a variety of different purposes. The aggregate database covers a wide range of force ratios and while emphasising 20th Century battles, has reasonable coverage back to 1600. It emphasises land battles, but includes one air campaign. Most report just initial and final values, but some time correlated data is included. Inevitably, there are issues which must be understood when attempting to reuse such data for purposes other than that for which it was compiled. Many of these have been previously studied [15] [16], including:

- potential bias in the battle narrative due to most accounts being written by the victor or for propaganda purposes,
- many reported results are qualitative or approximate,
- many reported results must be incorrect, including dispute over which side won,
- when determining force strengths should support or service personnel be included,
- when determining casualties should prisoners be included,
- how should force strength be obtained from numbers of participating staff, should some form of force scoring such as the Quantified Judgement Method be used,
- how should the effect of leadership, initiative, surprise, terrain and weather be included.

While some of the authors of the database components have attempted to address some of these issues [16], especially questions of how to determine force strength and casualties, there remains a question regarding the accuracy of much of the original reporting, especially for battles prior to the 19th Century. Nevertheless, the database reviewed here [15] is the best available source to carry out the proposed examination.

Hartley has argued that the individual datasets comprising the database are random samples, because they were independently derived [15]. This argument is similar to the inverse of Bootstrap sampling [17], which has been used to improve the accuracy of measures of sample statistical descriptors. A short account of sampling bias is given in Appendix A. Aggregation will improve the accuracy of statistical estimators, however, the outcome assumed by Hartley, that while the database is not a true random sample it can be treated as if it were effectively random, requires further consideration. It is difficult to avoid the conclusion that the database is little more than an aggregate of accidental sample databases.

The individual component databases are the product of the recursive application of the sub-sampling process. The population consists of all battles. This is first sampled to produce the set of all recorded battles. Many, especially smaller engagements, are never recorded. The requirement that both the initial and final values of forces strengths are known produces another sub-sampling stage to generate the set of all recorded battles with usable data. This sampling process also discriminates against smaller battles. Larger battles receive more attention and hence are more likely to have their attributes recorded. The individual databases are themselves samples of that sample.

Even if the final sampling process was random, the process of recording history generates a bias towards larger battles. By stratifying the aggregate database according to battle size, the lowest stratum containing the smallest battles can be seen to be the most affected. Together with the improved accuracy from aggregation, strata other than the lowest can be treated as if they were the product of random sampling. This conclusion will be tested using the database.

Previous work by the author [18] has shown that the initial and final strengths for both sides of a battle are related by:

$$\ln\left(\frac{x_0^2 - x_f^2}{y_0^2 - y_f^2}\right) = \alpha \ln\left(\frac{x_0}{y_0}\right) + \beta$$
$$\ln(HelmboldRatio) = \alpha \ln(ForceRatio) + \beta \qquad (21)$$

and that this relationship is a result of Lanchester's model for attrition (equation 1). Helmbold, and later Hartley [15], have shown that the data from an ensemble of different battles also follows equation 21.

Lanchester's Equations, and the proposed attrition equations (equation 18), describe the behaviour of a single system in time. However, the historical databases contain information about an ensemble of battles, each potentially with different values of attrition coefficients *a* and *b*. Should the results from such an ensemble follow the behaviour expected of a single system?

Hartley [15] has examined this issue at length. He considered several hypotheses, rejecting all save the conclusion that the relationship between the data from an ensemble of different battles was a direct consequence of the form of the equations governing the attrition process. In other words, the behaviour governing an individual battle was reflected in the behaviour of an ensemble of battles. Indeed, Helmbold's earlier work on the validation of Lanchester's equations using historical data [16] found this applies to a number of different parameters including the defender's advantage.

Using Hartley's conclusions, albeit more empirical than rigorous, and the random nature of the data samples for other than the lowest stratum, it is expected that the distribution of casualties during a battle (equation 18) should also be reflected in the distribution of casualties from an ensemble of battles. This conclusion will be supported, and less likely to be the result of artefact or data bias, if such patterns are only found in quantities affected by attrition (equation 18) and not in other quantities such as initial force strengths.

The solutions (equation 19) to Lanchester's Equations are monotonically decreasing and can be approximated as single exponential functions provided $|A| << |B|$. This approximation defines the region of validity for which log-normal solutions to equation 18 can be expected. It corresponds to the condition that the two sides are not significantly mismatched in their combat potential. Just as the process of data sampling has an inherent bias towards larger battles, it also contains a bias towards more "*even*" battles. An attacker that recognises it is outmatched has the option to use its initiative and not attack. Similarly, a defender that recognises its inferiority usually tries to improve its position before accepting battle. Protagonists appear more willing to accept battle when they are evenly matched and have a better expectation of a favourable outcome. Battles with smaller values for $|A|$ can therefore be expected to be over-represented in any data compilation. It is then reasonable to expect that the condition $|A| << |B|$ will apply to more data in the dataset, consistent with the requirement for battle casualty data to exhibit a log-normal distribution.

5.2 Results

The database contains a number of duplicate entries. One reason for this was to accommodate the different reporting of particular battles, including the identity of the winning side. The present work is not concerned with differentiating between winner and loser, which necessitates the removal of duplicate entries from the database.

Most previous work on frequency distribution of force sizes and casualties has only examined total combatants or casualties [16]. The present work examines the applicability of equation 18 for attrition modelling and so must consider each side separately. Hence each battle will contribute two data points to the analysis.

5.2.1 Force Size Distribution

The frequency distribution of initial force sizes was determined by dividing the range of force sizes into intervals of 1000 and counting the number of times a force strength from the database occurred in each interval. This is shown on a logarithmic scale in Figure 1.

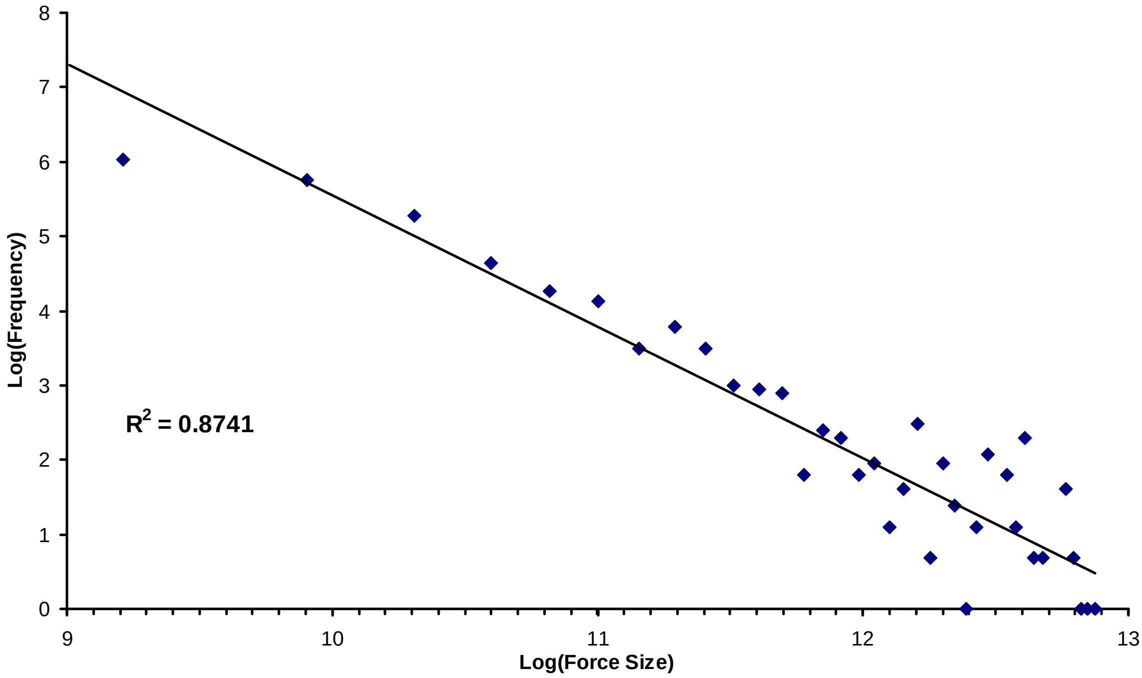

Figure 1: Force Size Distribution, and regression coefficient of determination

This is observed to follow an inverse power law relationship, which might have been expected on the basis of previous work dating back to Richardson [14]. He found that the frequency distribution of casualties in wars followed an inverse power law. Such fractal relationships exhibit scale invariance. Hence it is not unreasonable to find similar relationships for other similar quantities, such as the force size distribution for battles.

It should also be noted that the data progressively drops below the line of the linear relationship, obtained by regression analysis, for smaller force sizes. This is the behaviour

expected by the inherent data bias favouring larger battles from the application of sampling techniques described above.

The database contains some 1500 force size entries. Extrapolating from the best fit line, excluding those entries affected by bias, indicates that an unbiased database covering the same domain should be expected to contain at least 1800 entries if small size battles were not underrepresented.

It should also be noted that the distribution of initial force sizes does not exhibit any indication of the influence of a normal distribution. The completely different behaviour of the initial strength frequency and the casualty frequency supports the contention that such behaviour results from the attrition process and is not an artefact of the sampling or analysis procedure.

5.2.2 Casualty Distribution

The distribution of the natural logarithm of each side's battle casualties was determined by dividing the range of observed logarithm of casualty values into intervals of size 1, which is equivalent to the size for adjacent intervals having a ratio of 1.65. It results in an even spread of casualty values on a logarithmic scale which is necessary for the accurate representation of its distribution. The number of times the logarithm of the casualty value from the database occurred in each interval was then counted. This is shown in Figure 2.

The frequency distribution forms a bell shaped curve, but with considerable stochastic variability across the peak. This limits the ability to determine what form the distribution takes, in particular whether it is consistent with a normal distribution. Table 1 contains the summary statistics for the logarithm of casualties from the database.

Table 1: Log-Casualty Database Descriptive Statistics

Database Statistic	*Value*
Number of Entries	1498
Mean	7.485
Mode	8.006
Median	7.474
Standard Deviation	2.003
Skewness	0.105
Kurtosis	-0.153
Minimum Value	0
Maximum Value	13.693

The cumulative casualty distribution was formed by summing the number of occurrences with casualties greater than the specified value and is also shown in Figure 2. The cumulative distribution for occurrences greater than the specified value was chosen as it confines the effect of data bias to a few entries at the lower end of the scale instead of incorporating the bias in all the data points.

Table 1 indicates that the data is slightly skewed towards higher casualties, which is what is expected from our consideration of data bias issues. The previous section has indicated that the database should contain at least 1800 entries, if there was no bias against smaller battles. This bias can be compensated for by calculating the curve that the distribution would follow, given that number of entries, assuming a normal distribution and using values for the mean and standard deviation chosen from the historical data. Given the logarithmic nature of the scale, the additional entries can be assumed not to change the distribution mean by much. However, they will influence the standard deviation more. Figure 2 also shows the theoretical cumulative frequency distribution assuming that the logarithm of casualty values are normally distributed, for a total of 1800 entries, a mean of 7.5 and a standard deviation of 2.2.

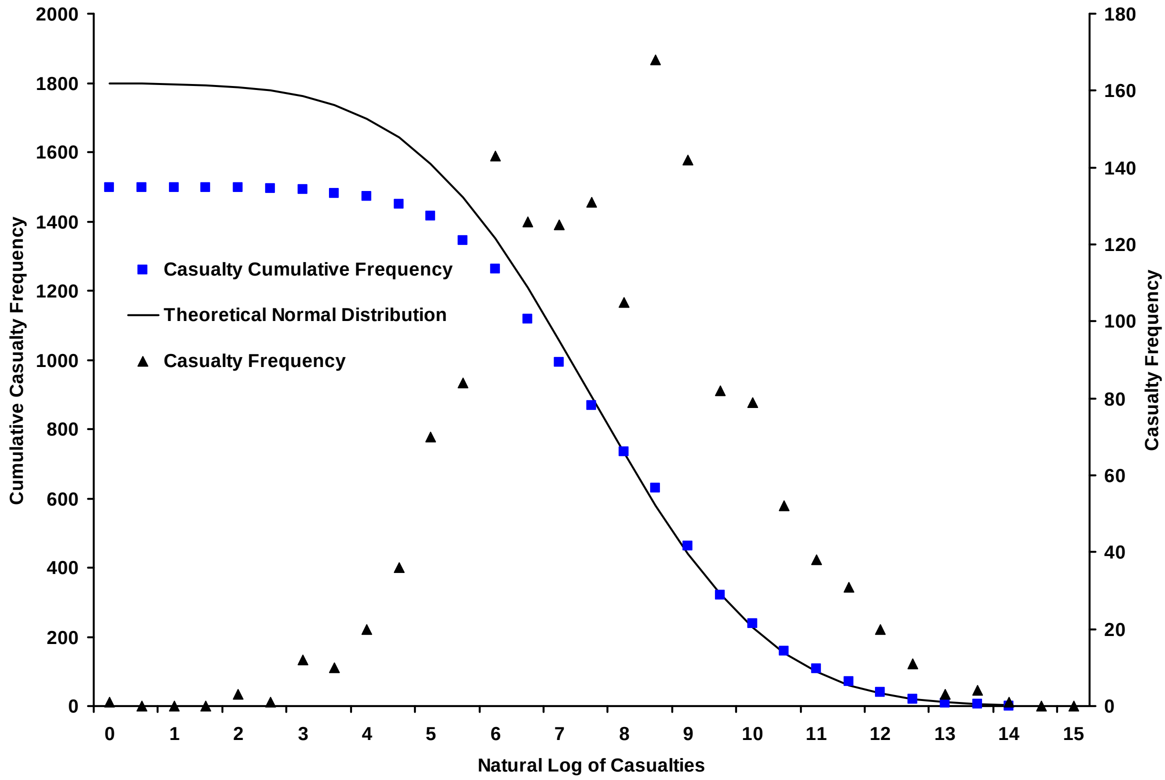

Figure 2: Log-Casualty Distribution, Casualty Cumulative Distribution, and Theoretical Cumulative Normal Distribution

The close agreement between the historical data and the theoretical distribution, except for the lowest database stratum where bias is expected to have produced under-representation, is apparent. The relationship between historical data and theoretical result can be observed more readily by looking at the correlation between the two sets of values.

Using standard statistical techniques [19], the correlation coefficient between the two sets of values, ignoring the lowest database stratum, was determined as 0.9965. This can be also be seen graphically by plotting the historical cumulative casualty distribution as a function of the theoretical expectation, as seen in Figure 3. Again ignoring the lowest stratum, regression analysis produces the best fit relationship shown, with a coefficient of determination of 0.9958.

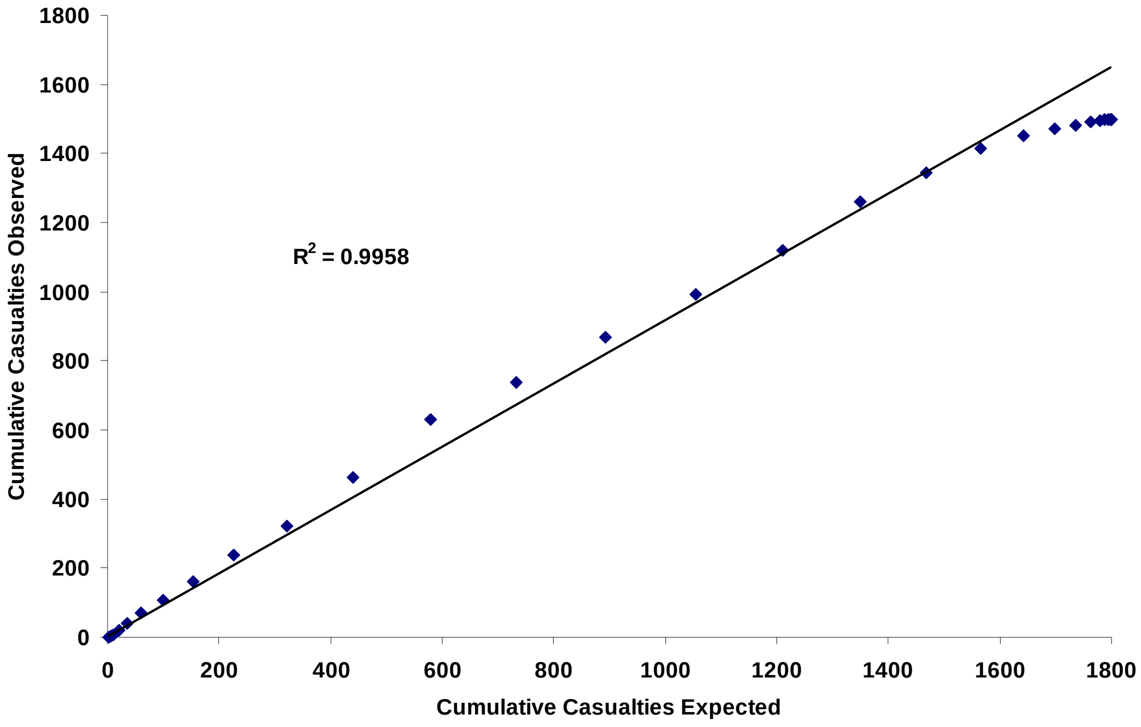

Figure 3: Force Size Frequency, and regression coefficient of determination

One standard interpretation [19] of the coefficient of determination is that 99.58% of the variability in the cumulative log-casualty frequency distribution observed in the historical record can be explained by the variation in a casualty model with a log-normal distribution, such as that proposed in equation 18.

This is consistent with the expectation from the hypothesis proposed in section 5.1.

6. Conclusions

Hartley's historical battle database [15], which previously has been used to validate the fractal nature of Lanchester's attrition equations by including spatial effects [3], has been used here to examine whether attrition is adequately modelled by a Markov process. It has been shown that the frequency distribution of battle casualties is consistent with that produced when attrition is modelled as an Ito process. The additional Wiener term can be regarded as representing the impact of the wider environment on attrition rates.

This battle database, and indeed all such databases, was shown to include an inherent bias which under-represents the number of small battles. While the effect of such bias was observed, by incorporating strata sampling concepts it is possible to confine the effects of such bias into a single stratum and a small number of data points, which can then be allowed for.

The shape of the distribution resulting from the assumption of a controlling Ito process was not observed in the initial force strength distribution. This supports the contention that such distributions result from the attrition process itself and are not artefacts of the sampling or analysis procedure.

The wider implications of the need to revise Lanchester's Equations by the inclusion of Wiener terms below representing the impact of the larger environment on attrition rates requires further investigation.

$$\begin{aligned} \frac{dx}{y} &= -a\,dt + \sigma dz_2, \quad x(0) = x_0 \\ \frac{dy}{x} &= -b\,dt + \varepsilon dz_1, \quad y(0) = y_0 \end{aligned} \tag{22}$$

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Appendix A: Bias in Sampling Theory

Sampling systems are used to obtain estimates of properties of the population being studied, and will be judged by how good the estimates obtained are. A good sampling system will, on occasions, give an estimate which is far from the true value, just as a poor system may, very occasionally, give an estimate very close to the true value. A system is better judged by the frequency distribution of the many estimates which are, or could be, obtained by repeated sampling. A good system would give a frequency distribution with small variance, and mean estimate about the same as the true value. The difference between the mean estimate and the true value is called the bias.

Bias may arise from a poor method of analysis, but more often from a poor choice of samples, or from the method whereby the measurement or counts are made or the samples are obtained. If the size of sample increased, or the data of two or more samples combined, then the bias will remain unaltered, but the variance will be reduced. Bias can normally only be detected and hence eliminated by careful examination of the whole sampling process from beginning to end.

The basic concept in all sampling is the random sample. A sample of objects from a population is random if all the members of the population have an equal chance of appearing in the sample. It is very important to remember that this applies to all members of the population, exceptional as well as typical members. If random numbers, or a similar randomizing process are not used, then it is likely that all individuals in the population will not have equal chances of appearing in the sample. If there is any correlation between the quantity being measured and probability of appearing in the sample, the result may be biased, perhaps strongly.

When sampling a heterogeneous population the precision achieved can be increased and the risk of bias reduced by dividing the population into sections, each relatively homogeneous, and sampling each section (or stratum) separately. Each stratum is then sampled independently, and estimates obtained for each. These can then be combined to give the estimate for the whole population. If entire groups of a heterogeneous population are excluded from a sample, there are no adjustments that can produce representative estimates of the entire population. However, if some groups are underrepresented and the degree of under representation can be quantified, then sample weights can correct the bias.

When the population being sampled is extensive or complex, the practical problems in taking a simple random sample are great, and the time taken for even a small sample may be large. The time required to obtain a sample of a given size may be greatly reduced by carrying out the sampling in two stages. First the complete population may be divided into a number of distinct primary units or subpopulations, and from these a sample is taken. From each of these sampled sub-populations a secondary sample, or subsample of individuals is taken.

Weakest of all sampling procedures, accidental sampling involves using what is available, and most convenient, as a sample pool.

Page classification: UNCLASSIFIED

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION
DOCUMENT CONTROL DATA

1. PRIVACY MARKING / CAVEAT (OF DOCUMENT)

2. TITLE Application of Black Scholes Complexity Concepts to Combat Modelling	3. SECURITY CLASSIFICATION (FOR UNCLASSIFIED REPORTS THAT ARE LIMITED RELEASE USE (L) NEXT TO DOCUMENT CLASSIFICATION) Document (U) Title (U) Abstract (U)
4. AUTHOR(S) Nigel Perry	5. CORPORATE AUTHOR DSTO Defence Science and Technology Organisation Fairbairn Business Park Department of Defence Canberra ACT 2600 Australia

6a. DSTO NUMBER	6b. AR NUMBER	6c. TYPE OF REPORT	7. DOCUMENT DATE
DSTO-TR-2318	AR-014-581	Technical Report	July 2009

8. FILE NUMBER	9. TASK NUMBER	10. TASK SPONSOR	11. NO. OF PAGES	12. NO. OF REFERENCES
2009/1038013/1	07/034	JOD LRR	16	18

13. URL on the World Wide Web	14. RELEASE AUTHORITY
http://www.dsto.defence.gov.au/corporate/reports/DSTO-TR-2318.pdf	Chief, Joint Operations Division

15. SECONDARY RELEASE STATEMENT OF THIS DOCUMENT

Approved for public release

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18. DSTO RESEARCH LIBRARY THESAURUS http://web-vic.dsto.defence.gov.au/workareas/library/resources/dsto_thesaurus.shtml

19. ABSTRACT
Lanchester's equations are commonly used as the basis for force-on-force combat models, even if only as a metamodel for a more complex combat simulation. This report examines whether attrition is adequately modelled by such Markov processes. It shows that the distribution of battle casualties is consistent with that obtained when attrition is modelled as an Ito process. The additional Wiener term can be regarded as representing the impact of the wider environment on attrition rates.

Page classification: UNCLASSIFIED